A CONTINUOUS DISTRICTING MODEL
APPLIED TO LOGISTICS DISTRIBUTION PROBLEMS

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The aim of districting problems is to get an optimized partition of a territory into smaller units, called districts or zones, subject to some side constraints such as balance, contiguity, and compactness. Logistics districting problems usually involve additional optimization criteria and constraints. Districting problems are called continuous when the underlying space, both for facility sites and demand points, are determined by variables that will vary continuously. Voronoi diagrams can be successfully used in association with continuous approximation models to solve location-districting problems. We discuss in the paper the application of non-ordinary Voronoi diagrams in logistics districting problems, particularly the Power Voronoi diagram, associated with a continuous demand approach, which allows for the introduction of physical barriers into the vehicle displacement representation, such as rivers, reservoirs, hills, etc.

Palavras-chaves: Logistics distribution, Districting, Voronoi diagrams, Optimization
1. Introduction

The vehicle routing problem (VRP) is the one of designing a set of routes from a central depot to various demand points, each having service requirements, in order to minimize the total distance. The total distance travelled is often substituted by cycle time or by a cost function. When customer demands or some other element of the problem are random variables, we have the stochastic vehicle routing problem (SVRP) (BASTIAN AND RINNOOY KAN, 1992). Common examples are stochastic demands and stochastic travel times. In addition, sometimes the set of customers to be visited is not known with certainty. The primary objective of such models is to find optimal tours, i.e. the best sequence of visits in order to minimize the total travelled distance, cycle time, or the total transportation cost, respecting, at the same time, service requirements (STEWART AND GOLDEN, 1983; BERTSIMAS, 1992).

The aim of districting problems, on the other hand, is to get an optimized partition of a territory into smaller units, called districts or zones, subject to some side constraints (HOJATI, 1996; MEHROTRA, JOHNSON AND NEMHAUSER, 1998; BOSKAYA, ERKUT AND LAPORTE, 2003). The constraints reflect a number of common sense criteria. One of them is to balance demand among districts. Furthermore, the resulting districts must be contiguous and geographically compact (WILLIAMS, 1995). Logistics and transportation districting problems usually involve additional optimization criteria and constraints. In general, apart from the basic balance, contiguity, and compactness principles, there is not a set of general criteria that are common to all districting problems. Districting problems are called continuous when the underlying space, both for facility sites and demand points, are determined by one or more variables that will vary continuously.

Districting problems are associated with a number of practical applications. Political districting, in which one is interested in drawing of electoral district boundaries, has received much attention in the literature (HOJATI, 1996; MEHROTRA, JOHNSON AND NEMHAUSER, 1998; BOSKAYA, ERKUT AND LAPORTE, 2003, WILLIAMS, 1995). School districting (SCHOEPFLE, AND CHURCH, 1991) and police districting (D’AMICO et al., 2002) are two other areas of research interest. In addition, the literature presents articles on the design of sales territory (ZOLTNERS AND SINHA, 1997), as well as emergency, health-care, and logistics districting. Among the latter, we mention the balanced allocation of customers to distribution centers (Zhou, Min AND Gen, 2002) and the design of multi-vehicle delivery tours (LANGEVIN AND SAINT-MLEUX, 1992; NOVAES, DE CURSI AND GRACIOLLI, 2000).

Continuous approximations to districting problems are based on the spatial density and distribution of the demand rather than on precise information on every demand unit. It allows for simple, yet robust models that are useful when planning a new service or the expansion of an existing one (LANGEVIN, MBARAGA AND CAMPBELL, 1996; NOVAES, DE CURSI AND GRACIOLLI, 2000; DASCI AND VERTER, 2001).

The association of continuous approximation techniques with Voronoi diagrams opens the way to solve a number of real-life districting problems. In particular, the use of non-ordinary Voronoi diagrams to solve logistics and transportation problems has been reported in the literature (OKABE, BOOTS AND SUGIHARA, 1995; SUZUKI AND OKABE, 1995; GALVÃO et al., 2006; NOVAES et al., 2009). Boots and South (1997) used a multiplicatively weighted Voronoi diagram approach for modeling retail trade areas. Galvão
et al. (2006) defined a multiplicatively weighted Voronoi diagram model to solve an urban freight distribution problem. The utilization of non-ordinary Voronoi diagrams in logistics districting problems, associated with a continuous demand approach, also allows for the introduction of physical barriers into the model (NOVAES et al., 2009). This is an important property because it permits to treat problems with obstacles imposed by thoroughfares, highways, rivers, reservoirs, hills, etc.

The purpose of this paper is to apply a continuous districting model to a distribution logistics problem, combining a power Voronoi diagram approach with an optimization algorithm.

2. Continuous approximation

A large part of discrete districting problems can be converted into problems involving continuous functions, with good practical results. With such formulation, demand potentially arises at any point in a plane and feasible locations of facilities are equally any point in a plane. Continuous demand approximation models are usually based on the spatial distribution variables rather than on precise information on every servicing point (LANGEVIN, MBARAGA AND CAMPBELL, 1996; DASCI et al., 2001). A number of urban districting problems have been solved with this approach (NEWELL AND DAGANZO, 1986a; NOVAES, DE CURSI AND GRACIOLLI, 2000; GALVÃO et al., 2006; NOVAES et al., 2009). Before choosing an appropriate algorithm to solve a districting problem with this approach, it is necessary to represent the data in a continuous format.

In the application analyzed in this paper, the variables to be continuously represented can be the number of visiting points in a sub-region, the demand intensity at point (x, y), or other attribute of interest. Methods for the construction of a regular continuous approximation to a function may be found in the literature. Usually a good approximation is attained with a bi-quadratic spline (BOOR, 2001) combined with a finite element discretization of the region under analysis (BATHE, 1982). These techniques were adopted in this research since the use of finite elements saves computational time. But the method may be implemented with any kind of finite element mesh by using the appropriate weights in order to evaluate the function to be approximated. For more details, the reader is referred to Galvão et al. (2006).

2.1. Guidelines for district design

Districting problems are associated with one or more ‘activities’ or ‘operations’ to be performed within the served region and considering each district individually. From a planning point of view, districting should be performed at the strategic or tactical level, while the detailed optimization of activities should be performed at the operational level. In another words, districting should involve a more global view and is usually related to the managerial and administrative levels. Thus, district borders should not change too frequently but, instead, should be modified only when major system and demand changes take place (MUYLDERMANS et al., 2002).

In general, the methods for solving districting problems follow a sequence of procedures (FLEISCHMANN et al., 1988):

- Definition of one or more activity measures, or queries (OKABE, BOOTS AND SUGIHARA, 2000);
- Set up a function (or functions) to represent the ultimate objective of the districting process;
- Set up a number of constraints to be respected in the districting process;
- Solve the districting problem with an appropriate algorithm.
In some cases, the underlying query is quite simple. For example, in a simple political districting problem one may wish to partition the region into a number of districts such that each district has almost the same number of voters. The basic query is then “compute the total number of voters in each district”, and the representing function is the integral of the density of voters over the district area. If \( V_i \) is the total number of voters in district \( i \), the associated balance constraints would be

\[
|V_i - V_j| \leq \varepsilon, \quad i = 1, 2, \ldots, m, \text{ and } i \neq j,
\]  

where \( m \) is the number of districts and \( \varepsilon \) is a tolerance level.

In a more complex case, the query may be “find the maximum distance from the depot to any client premise within the district”, for which the optimizing function could be the sum of the squared deviations of such distances, to be minimized, i.e.

\[
\min S = \sum_{i=2}^{m} (d_i^{(F)} - d_{i-1}^{(F)})^2,
\]

where \( d_i^{(F)} \) is the maximum distance from the depot to any client premise within the district \( i \).

Some transportation and logistics problems involve more complex queries because the operations to be performed in a district depend on a number of endogenous and exogenous variables. Each problem has its own set of queries, associated with functions and constraints, but some general recommendations should be kept in mind:

- As pointed out by Muyldermans et al. (2002), most districting problems require a multi-criteria approach, and the queries and related functions must reflect this requisite when necessary. Fleischmann et al. (1988), for instance, in order to measure salesman workload to perform a marketing effort, adopted a compound score that takes into account the sales revenue and the frequency of visits to retailer customers in each district;
- Since districting should preferably be performed at the strategic or tactical level, the function (or functions) to represent the ultimate objective of the districting process should not involve too many variables and detailed modeling. More effort should be applied to reduce computing time due to the large number of iterations when running such models;
- Balance among districts must be appropriately represented by one or more constraints in the model;
- In order to avoid violating the requirement that the served units in each district are contiguous and each district is geographically compact, the model must incorporate appropriate heuristics to satisfy such requisites. The Voronoi diagram approach facilitates this task since, if the Voronoi diagram type is well chosen and well employed, contiguity and compactness conditions are naturally respected.

3. Voronoi diagrams

In this section it is presented a summary of the basic concepts and properties of Voronoi diagrams that will further be used to solve a logistics districting problem. Voronoi diagrams comprise a vast subject, and the reader is referred to Aurenhammer (1991) and Okabe, Boots and Sugihara (2000) for more details. Although Voronoi diagram generators can be points, lines, circles, or areas of diverse shapes, our application deals with point generators only. In fact, logistics problems usually involve “point-like facilities” such as depots, client premises, trucks, etc. Presently, Voronoi diagrams are extensively used in computational geometry, computer graphics, robotics, pattern recognition, games, etc (AURENHAMMER, 1991).
The basic concept of Voronoi diagrams is quite simple: given a finite set of distinct and isolated points in a continuous space (generator points), we associate all locations in that space with the closest member of the point set (OKABE, BOOTS AND SUGIHARA, 2000).

In the simplest case of Voronoi diagram the distances from any point to the generator points are represented by the Euclidean norm. The edges of the resulting Voronoi polygons in a plane are line segments. There are situations, however, when the Euclidean distance does not represent well the attracting process. For instance, suppose that the six generator points exhibited in Figure 1 are retail stores selling the same kind of product. Assume further that, in addition to distance, the attraction of such stores depends on a set of features, leading to the weighting coefficients shown in Figure 1. In order to take these elements into account, several kinds of weighted Voronoi diagrams have been developed. These diagrams use a family of weights \( w = (w_1, w_2, ..., w_m) \) such that the dominance region increases with the weight \( w_i \). For instance, multiplicatively weighted planar Voronoi diagrams correspond to

\[
\mu_i(X, P) = \frac{1}{w_i} \|X - P_i\|, \tag{3}
\]

where \( w \) is a family of strictly positive weights. In the case with only two generator points, the locus of the points \( X \) satisfying (3) is the Apollonius circle (OKABE, BOOTS AND SUGIHARA, 2000), except if \( w_1 = w_2 \), when the bisector becomes a straight line. Fig. 1 shows an example of MW-Voronoi diagram for the weights there indicated. In general, a MW-Voronoi region is a non-empty set and need not be convex, or connected; and it may have holes (OKABE, BOOTS AND SUGIHARA, 2000).

![Figure 1 – An example of a multiplicatively-weighted Voronoi diagram](image)

Analogously, the additively weighted Voronoi diagram is represented by

\[
\mu_i(X, P) = \|X - P_i\| - w_i. \tag{4}
\]

Here, the sign of \( w_i \) is not restricted. The combination between additive and multiplicative weights leads to the compoundly weighted Voronoi diagram, which is associated to

\[
\mu_i(X, P) = \frac{1}{w_{i1}} \|X - P_i\| - w_{i2} \tag{5}
\]
We recall that the sign of $w_i$ is not restricted. In this case, the boundary of the dominance region is a fourth-order polynomial function, and its shape is fairly complex. The power Voronoi diagram corresponds to

$$\mu_i(X, P_i) = \|X - P_i\|^2 - w_i$$

In this case, only positive values of $w_i$ are usually used. The line segment connecting $P_i$ and $P_j$ is a straight line perpendicular to the line segment $P_j - P_i$. An important property of power Voronoi diagrams, useful in applications, is that the resulting Voronoi polygons are always convex. Power Voronoi diagrams are especially useful to solve districting problems with barriers, as in the case described in Section 4.1. When solving Voronoi diagram problems with obstacles, the Euclidean norm is not acceptable. If an obstacle lies on the line linking an origin and a destination, it is not possible to traverse it straight. Instead, a detour around the obstacle must be taken.

Following Okabe, Boots and Sugihara (2000), let us consider a generator set $P$ and a set of $c$ closed regions $O = \{O_1, \ldots, O_c\}$ ($1 \leq c < \infty$). The set $O$ represents a set of obstacles that are not traversable. These obstacles are assumed not to overlap each other and points of $P$ are not allowed to lie within the obstacles (Figure 2a). Furthermore, each obstacle is assumed to be connected and with no holes. For computational convenience it is assumed that $O_i$ ($i = 1, \ldots, c$) is a polygon, but it is not assumed that $O_i$ is necessarily convex. Line segments are also accepted as obstacles. The visibility-shortest-path distance between a generic point $X$ and a generator point $P_i$, expressed as $d_{sp}(X, P_i)$, is obtained considering all possible continuous paths connecting $X$ and $P_i$ that do not traverse obstacles. A visibility polygon with respect to $P_i$, and denoted by $Vis(P_i)$, is the set of points that are visible from $P_i$.

An example is shown in Figure 2a, where the visibility polygon with respect to the generator point $P_1$ is indicated by the hatched area. To compute the visibility-shortest-path distance between a point $X$ and a generator point $P_1$ one uses the correspondent visibility graph, which is formed by all possible paths connecting $X$ and $P_1$ (Figure 2b). On the visibility graph one solves the classical shortest-path problem with the aid of an appropriate algorithm such as, for example, the well-known Dijkstra method. For the example of Figure 2b, the shortest-path between $P_1$ and $X$ is $P_1 \rightarrow B \rightarrow F \rightarrow X$. 


4. An urban distribution problem with a geographical barrier

Most logistics distribution and collecting problems involve spatial variables associated with operational and economic elements, such as routing, vehicle capacity, vehicle costs, servicing times, etc. As pointed out by Muyldermans et al. (2002), districting associated with transportation and logistics problems should be performed at the strategic and tactical level, whereas routing should be performed at the operational level. In other words, districting involves a more global view and is often related to the managerial and administrative levels, while routing processes are more detailed and linked to day-to-day operations.

In many applications, locations are characterized by Cartesian coordinates and distances are estimated by an $L_2$ (Euclidean) metric, corrected by a routing factor (GALVÃO et al., 2006). Additionally, in one-to-many distribution and collecting problems with multiple tours (DAGANZO, 1996), an idealized dense ring-radial network pattern is frequently adopted as a theoretical modeling basis. In such cases, the ideal configuration of the districts should be wedge-shaped and elongated toward the depot (NEWELL AND DAGANZO, 1986a). While interesting from a theoretical point of view, this approach is not readily applicable to real life situations. For more generic metrics, in fact, the optimal orientation of the districts and its shape are not obvious. The non-definition of the real local network metric makes the ideal shape of the zones unclear. Furthermore, since the real transportation infrastructure usually presents a coarse network of roads with varying velocity, the ideal orientation of the districts is also unclear (NEWELL AND DAGANZO, 1986b). Some computational tests (GALVÃO et al., 2006) have indicated that the adoption of a Voronoi diagram partitioning process changes the resulting total distribution cost only marginally when compared with the
corresponding ring-radial results, and therefore the Voronoi formulation may be used as an adequate districting approximation in a variety of practical applications.

In this section we describe and discuss one application of a generalized Voronoi diagram to a logistics distribution problem with barriers in order to illustrate the possibilities of the method.

4.1. Problem description

The utilization of generalized Voronoi diagrams in logistics districting problems has some advantages. The fitting process, for instance, leads to more equalized load factors among the districts, meaning the vehicles assigned to the zones will show more balanced utilization levels. This happens because the generalized Voronoi diagrams have more degrees of freedom when searching for the district contours when compared to the wedge-shaped, geometrical partitioning scheme. Furthermore, as mentioned, the resulting total distribution cost is only marginally affected by such an approximation (GALVÃO et al., 2006). Additionally, the utilization of an appropriate Voronoi diagram approach opens the way to solve districting problems with geographical barriers imposed by thoroughfares, highways, rivers, reservoirs, parks, steep hills, etc.

The problem concerns an urban distribution service covering a specific area. The objective is to define the number of districts and their boundaries to be assigned to the delivery vehicles in order to: (a) minimize total daily delivery costs; (b) balance the distribution effort among the vehicles, and (c) respect capacity constraints. Additionally, the resulting districts must be contiguous and geographically compact (MEHROTRA, JOHNSON, AND NEMHAUSER, 1998). A homogeneous fleet of $m$ delivery vehicles is assumed and each vehicle is allocated to a district, performing a complete cycle per working day from a central depot, in a one-to-many distribution scheme (DAGANZO, 1996). The served urban region $\mathcal{R}$ may be of irregular shape and has an area $A$. The density of servicing points varies over $\mathcal{R}$ but is nearly constant and Poisson distributed over distances comparable with a district size (NEWELL AND DAGANZO, 1986a). The demand is formed by: (a) the total of number servicing points, (b) the average weight of cargo delivered at each client’s location (kg), and (c) the mean stopping time per visiting point (min).

Galvão et al. (2006) developed a multiplicatively weighted Voronoi diagram model to solve this problem with no geographical barriers. That same problem was extended to the situation in which geographical obstacles restrain districting limits (NOVAES et al., 2009). In this paper the obstacle is represented by a freeway, with crossing points spaced along the road (viaducts and underpasses). Those crossing points are congested most of the time. As a practical consequence, logistics operators do not design delivery districts covering areas situated in both sides of the freeway. This practical situation is handled, in the model, with the introduction of one obstacle represented by the mentioned freeway (Figure 3). The problem was solved with a power Voronoi diagram formulation associated with the visibility-shortest-path metric (Section 3).

4.2. Vehicle cycle modeling

Vehicle travel distances within the districts are assumed to be approximately represented by a Euclidean metric, with the real distances being estimated with the aid of a mathematical function (DAGANZO, 1984), usually a multiplying route factor greater than unit. Following Stein (1978), the expected distance $E[DZ]$ travelled by a vehicle within a district of area $A$, and $n$ visiting points, can be approximated as
where $\delta = n/A$ is the density of points over $A$. Expression (7) can be applied to most metrics (Novaes and Graciolli, 1999) and presupposes that the points are uniformly and independently scattered over the area, and the district is fairly compact and fairly convex. The coefficient $k_0$ can be expanded into two multiplicative factors (NOVAES AND GRACIOLLI, 1999). The first one depends solely on the adopted metric and routing strategy. The second factor is a corrective coefficient (route factor) reflecting the road network impedance.

The vehicle starts from the depot, goes to the assigned district, does the delivery or pick up, and comes back to the depot when all the visits are completed, or when the maximum allowed working time per day is reached, whichever occurs first. This complete sequence makes up the vehicle cycle. In some practical circumstances more than one tour per day can be assigned to the same truck. This implies extra line-haul costs, but depending on the cargo characteristics, vehicle size restrictions, and other factors, multiple daily tours per vehicle might sometimes be appropriate. For the sake of simplicity, we assume that the vehicles perform just one cycle per day. The model can be easily modified to take into account multiple daily cycles.

The total cycle length $D$ is the sum of the line-haul distance (either way) and the local travel distance given by (7). The total cycle time $T$, on the other hand, is the sum of the line-haul time, the local travel and the total handling time. The latter is the sum of the times spent in delivering the cargo at the customer’s locations. The expected value of $T$ for a generic district is

\[
\bar{T} = E[T] = \frac{2}{v_L} E[D_{LH}] + \frac{k_0 \sqrt{A n}}{v_Z} + p n E(t_S)
\]

(8)

where $E[D_{LH}]$ is the expected line-haul travel distance (one way) from the depot to the district, $v_L$ is the average line-haul speed, $v_Z$ is the average local speed, $p$ is the probability that a customer be visited, and $E(t_S)$ is the expected stop time spent in one delivery (Novaes and Graciolli, 1999).

Assuming statistical independence of the elements which form the cycle time, the variance of $T$ is given by

\[
\text{var}[T] = \sigma_T^2 = 2 \text{var}(t_h) + \text{var}(t_z) + p n \text{var}(t_S),
\]

(9)

where $t_h$ is the line-haul travel time (one way), and $t_z$ is the local travel time. Using the central limit theorem, $T$ can be represented by the normal distribution $T \sim N(\bar{T}, \sigma_T)$. We assume that the cycle time cannot exceed a maximum of $H_0$ working hours per day, imposed by labor restrictions and company policies. Let $\xi \sim N(0,1)$ be the unit normal variate. Adopting a 98 percentile (mono tail distribution), $\xi = 2.06$, and thus

\[
g_1 = \bar{T} + 2.06 \sigma_T \leq H_0
\]

(10)

is a restriction that must be respected. Let $E[u]$ and $\sigma_u$, on the other hand, be respectively the mean and the standard deviation of the quantity $u$ of product delivered per visiting point in the generic district. Then, assuming statistical independence of the customer’s demands, the
expected value and the variance of the total vehicle load $U$ for one tour in the district is given by

$$
\bar{U} = E[U] = p n E[u] \quad \text{and} \quad \text{Var}[U] = \sigma_u^2 = p n \sigma_u^2 .
$$

(11)

According to the central limit theorem, $U$ can be represented by the normal distribution $U \sim N(\bar{U}, \sigma_U)$. If $W$ is the truck capacity, and adopting a 98% monotail percentile, another restriction that must be respected is

$$
g_2 = \bar{U} + 2.06 \sigma_U \leq W .
$$

(12)

In place of restrictions (10) and (12), two loading factors assigned to district $i \quad (i = 1,2,\ldots,m)$ are used with the same objective, the first taking into account time utilization and, the second, vehicle capacity utilization

$$
\phi^{(T)}_i = \frac{g_1(i)}{H_0} \leq 1 \quad \text{and} \quad \phi^{(U)}_i = \frac{g_2(i)}{W} \leq 1 , \quad (i = 1,2,\ldots,m) .
$$

(13)

The load factor for district $i$ is the largest of $\phi^{(T)}_i$ and $\phi^{(U)}_i$

$$
\phi_i = \max\{\phi^{(T)}_i, \phi^{(U)}_i\} , \quad (i = 1,2,\ldots,m)
$$

(14)

Thus, the balancing criterion is to equalize load factors among the $m$ districts is

$$
|\phi_i - \phi_j| \leq \varepsilon \quad (i, j = 1,\ldots,m) ,
$$

(15)

where $\varepsilon > 0$ is a small tolerance factor.

### 4.3. Defining the number of districts

Assuming an initial value for $m$ and applying the model, it will be necessary to increase $m$ if the resulting values of $\phi_i \quad (i = 1,2,\ldots,m)$ are greater than one. Conversely, if the resulting load factors are too low, the value of $m$ should be reduced. This process continues until one gets a suitable solution respecting (15). Since the computing process takes time, it is recommended to choose a better estimate of $m$ to be initially used in the model.

Let $Q_R$ be the total quantity of cargo carried per day in region $\mathcal{R}$. If $W$ is the cargo capacity of a vehicle, a rough estimate of $m$ is

$$
m_U = \frac{Q_R}{W} .
$$

(16)

With regard to cycle-time restriction, a rough estimate of $m$ is given by the following formula (Novaes et al, 2009)
where $A_R$ is the area of region $\mathcal{R}$, $N_R$ is the total number of delivery points in $\mathcal{R}$, $\bar{\tau}$ is the average stopping time spent in one client, and $p$ is the probability a customer is visited in the tour. If the region $\mathcal{R}$ is approximately circular and the depot is fairly centralized, the average distance $D^{(L)}$ may be assumed to be (Novaes et al, 2009)

$$D^{(L)} \approx 2 \sqrt{\frac{A}{3\pi}},$$

(18)

Where $A$ is the average area of one district. Applying (16) and (17) one estimates $m_T$ and $m_U$, respectively. A good initial approximation for the number of districts $m$ is the largest value of $m_T$ and $m_U$. Applying the model and analyzing the resulting values of $\phi_i$ ($i=1,2,...,m$) in a recursive way, one may change the value of $m$, running the model again until an acceptable solution is obtained.

4.4 The iterative process

The districting problem with obstacles is solved with a power Voronoi diagram

$$\mu_i(X,P_i)=[d_{sp}(X,P_i)]^2 + w_i, \quad (i=1,2,...,m),$$

(19)

where $d_{sp}(X,P_i)$ is the visibility-shortest-path distance between point $X$ and the generator point $P_i$ (Section 3). Let $k$ represent the stage of the iterative process. At each stage the weights $w_i^{(k)}$ ($w_i^{(k)} \geq 0, \quad i=1,...,m$) are modified according to the following convergence rule

$$w_i^{(k)} = w_i^{(k-1)} + v_i^{(k-1)}$$

(20)

where $v_i^{(k-1)}$ is given by

$$v_i^{(k-1)} = \frac{\phi_i^{(k-1)} - \bar{\phi}^{(k-1)}}{d}, \quad d > 0$$

(21)

where $\bar{\phi}^{(k-1)} = \frac{1}{m} \sum_{i=1}^{m} \phi_i^{(k-1)}$ and $d$ is a control parameter. The value of $d$ is changed empirically in order to control the convergence of the model. Since $d > 0$, the weight $w_i^{(k)}$ of district $i$ will increase if $\phi_i^{(k-1)}$ is greater than the mean $\bar{\phi}^{(k-1)}$. Putting $w_i$ with a positive sign in (19), the dominance region of $P_i$ (the district area) will decrease (OKABE, BOOTS AND SUGIHIRA, 2000), tending to lead to a more balanced solution. If $\sigma^{(k-1)}$ is the standard deviation of the observed $\phi_i^{(k-1)}$ ($i=1,2,...,m$), the value of $d$ is chosen as to guarantee a decreasing sequence of $\sigma^{(k)}$

$$\sigma^{(1)} > \sigma^{(2)} > ... > \sigma^{(k)}$$

(52)
At stage $k$ the iterative process may terminate, in accordance to (15), if

$$
\phi^{(k)}_{\text{Max}} - \phi^{(k)}_{\text{Min}} < \varepsilon,
$$

where $\varepsilon > 0$ is a small tolerance factor. For $k = 1$, the weights $w_i^{(1)}$, $i = 1, \ldots, m$ are set equal to zero, leading to an ordinary Voronoi diagram configuration.

At each stage of the process the power Voronoi diagram is constructed with an appropriate algorithm (NOVAES et al., 2009), and the resulting relevant attributes are computed, in special, the center of mass of each district. The center of mass is related here to the concentration of delivery points within the district, since the number of stops is the prevailing variable when dimensioning this kind of service. The centers of mass of the districts are then taken as the generator points of the Voronoi diagram for the next stage of the iterative process.

4.5. Results and conclusions

We have presented a method to solve a logistics districting problem in which the displacement of vehicles is restricted by geographical barriers, employing a Power Voronoi diagram algorithm. The resulting Voronoi tessellation for this example resulted into 57 balanced districts, as shown in Figure 3. The resulting district contours are smooth and closer to the configuration contours encountered in practical situations, when compared to the traditional wedge-shape formulation.

The resulting partition of the region led to more balanced time/capacity utilization (load factors) across the districts. Considering the coarse underlying road-network approximation (Euclidean metric associated with a route factor), and the fact that a small error in the parameters tend to give only a much smaller increase in the cost, we conclude that the Power Voronoi diagram fitting process is a valid methodology to solve a number of practical distribution districting problems.

Apart from obtaining smoothed district partitions, the utilization of Voronoi diagrams opens the possibility for further exploring some of its properties in order to get better approximations to real-world problems. In special, Voronoi diagrams allow for the introduction of physical obstacles into the model, as shown in the present work. This kind of
situation occurs frequently in urban distribution problems, with obstacles imposed by thoroughfares, highways, rivers, reservoirs, hills, etc.

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